

Lecture 14: The Carnot cycle

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- The four steps of the Carnot cycle
- The efficiency of the Carnot cycle
- A real life Carnot cycle
- Our first look at entropy
- Advanced – The (Re)definition of temperature
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Hero's Fountain Returns



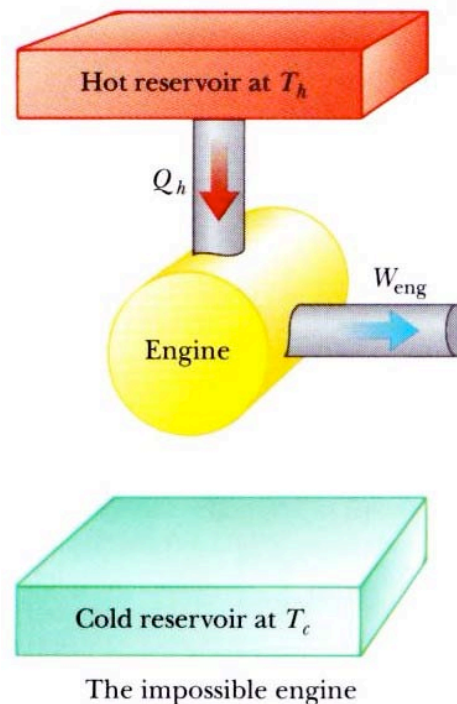
Ned Western has claimed the prize offered for a functioning, home-built Hero's Fountain. Well done!



The impossible engine

Kelvin-Planck statement of 2nd law: *No process is possible whose sole result is a heat flow Q out of a reservoir at a single temperature, and the performance of work W equal in magnitude to Q .*

In effect, this statement of the second law says that it is impossible for our generalised engine to look like...



Which is a shame, because this represents a 'perfect' engine with 100% efficiency.



The Carnot cycle

Given that, an interesting question to ask is: If we put all engineering and practical issues aside, what is the most efficient engine that you can make as far as physics is concerned?

In 1824, a French engineer named Sadi Carnot became interested in this question. The result was a theoretical engine called the Carnot engine that operates in a cycle consisting of two isothermal processes and two adiabatic processes, as we will soon see and analyse in detail. What is remarkable about this engine is that **it represents the most efficient engine possible** because it is a perfectly reversible cycle.

Unfortunately, as a practical engine, the Carnot engine is rather useless. Actually it's impossible to get it to work at all because any real process is not perfectly reversible. Still, on paper at least, this is the most efficient engine you can possibly make, which makes it very useful.



Carnot's Theorem

- Still, on paper at least, this is the most efficient engine you can possibly make. Carnot realised this, and this led to what is known as Carnot's theorem:

No real (irreversible) heat engine operating between two energy reservoirs can be more efficient than a Carnot (reversible) engine operating between the same two reservoirs.

The corollary of Carnot's theorem is that:

All Carnot (reversible) engines working between reservoirs at the same temperature have the same efficiency.

Note that I've put reversible in brackets in those statements because although Carnot originally referred to Carnot engines in his work, Clausius later showed that *any* reversible cycle is composed of a finite number of Carnot cycles, as we'll see in lec. 21 next week, making this a much more general result about reversible engines having superior efficiency to irreversible engines.



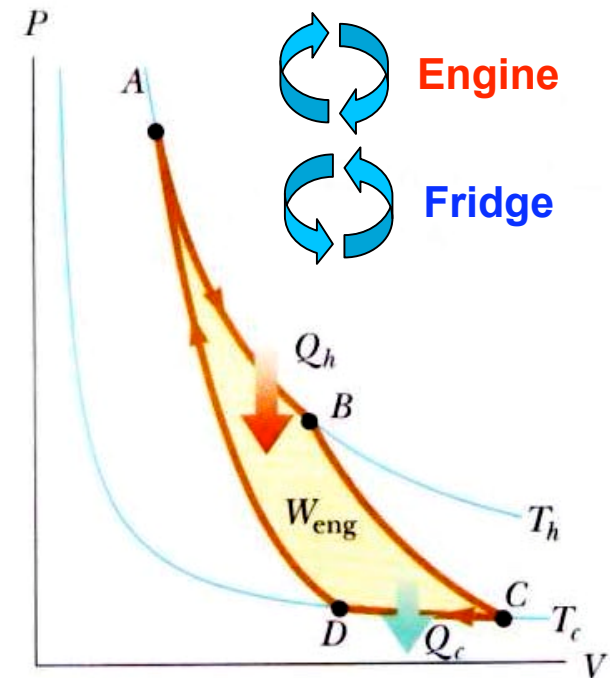
Proof of Carnot's Theorem

- Let's try and prove the validity of Carnot's theorem. Again we will use the same tactic of proving that the converse is impossible. So let's suppose we can make a 'super-engine', one that is more efficient than a Carnot engine, so it will have an efficiency $e > e_c$, where e_c is the efficiency of a Carnot engine.

The absolutely critical feature of this proof is that the Carnot engine is reversible – so if we go around the cycle one way (**clockwise**), we put Q_h in and get $W = e_c Q_h$ out with $Q_c = (1 - e_c)Q_h$ waste heat sent to the cold reservoir.

And if we go around the cycle the other way (**anticlockwise**), we get a Carnot refrigerator which takes $Q_c = (1 - e_c)Q_h$ and with the input of work $W = e_c Q_h$ provides Q_h to the hot reservoir.

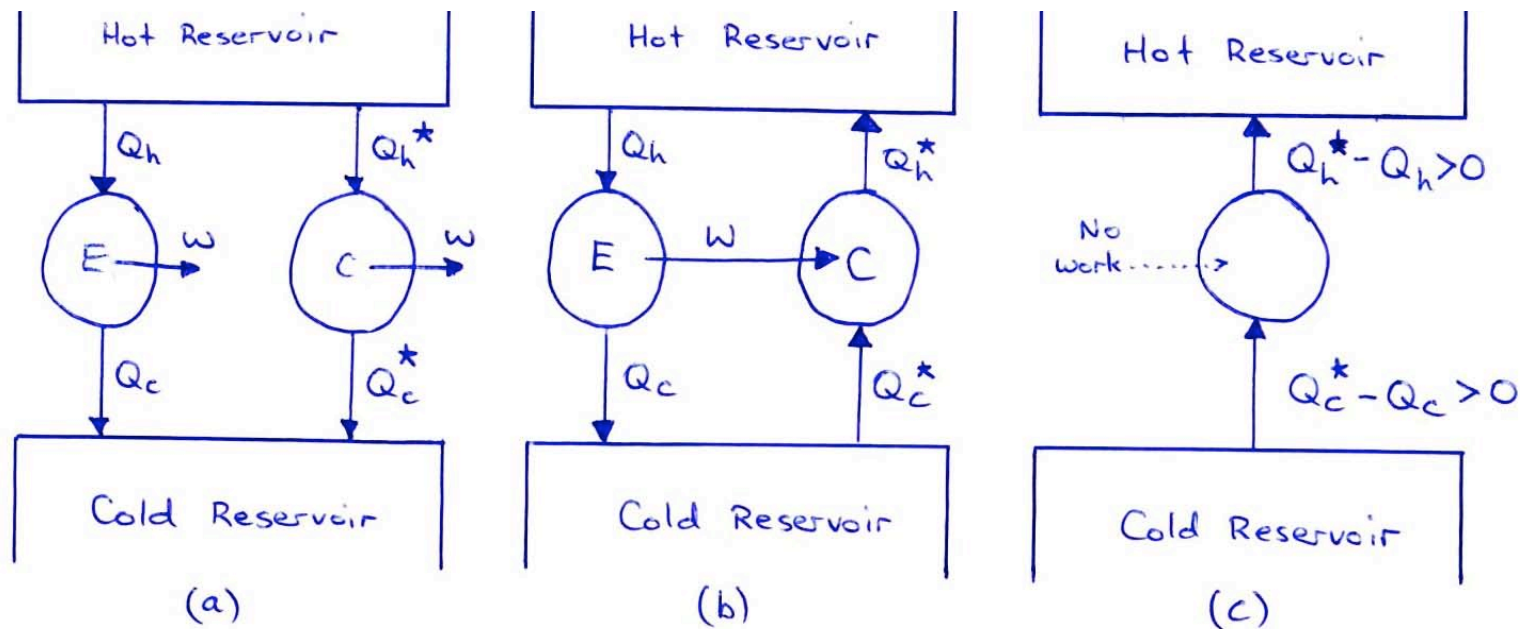
You can't do this with an irreversible engine – the cycle and its reverse cycle are not the same (we will see this more clearly later for adiabatic processes when we introduce entropy).



Proof of Carnot's Theorem

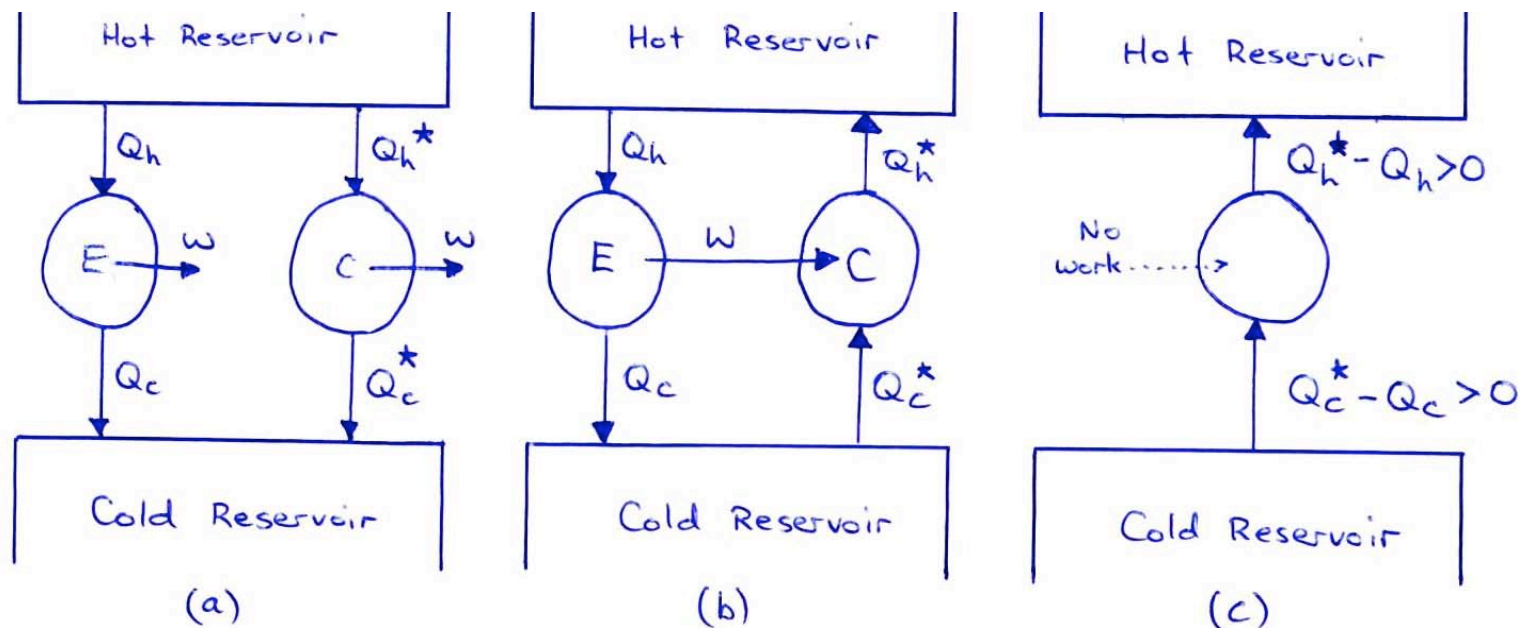
- Let's first consider Fig. (a) where we have our Carnot engine (with e_c , Q_h^* , W^* and Q_c^*) and our super-engine (with e , Q_h , W and Q_c), and run them so that they produce the same work $W = W^*$.

If $e > e_c$, then $W/Q_h > W/Q_h^*$, and so $Q_h < Q_h^*$ – in other words, because our super-engine is more efficient, it needs less heat input to produce the same work. Furthermore $Q_h = Q_c + W$ and $Q_h^* = Q_c^* + W$ and so we can take this further and write $Q_c < Q_c^*$ – in other words our super-engine also produces less waste heat because less input heat came in to create the work W .



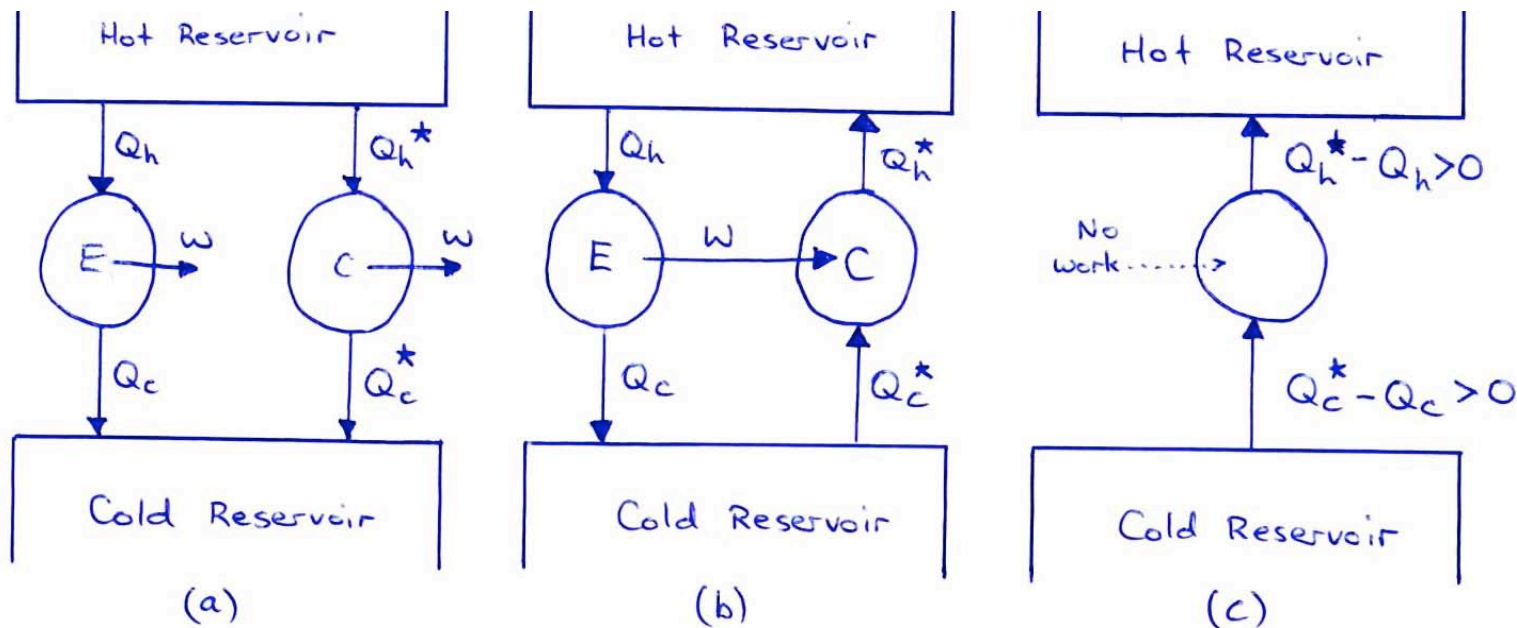
Proof of Carnot's Theorem

- Now that we have our terms figured out, in Fig. (b) we use our super-engine to drive the Carnot engine backwards, as a Carnot refrigerator. So our super-engine takes Q_h from the hot reservoir, puts Q_c into the cold reservoir and feeds work W into the Carnot refrigerator, which then takes Q_c^* from the cold reservoir and puts Q_h^* into the hot reservoir.



Proof of Carnot's Theorem

- In Fig. (c) we show the net effect of this combination, it acts as a refrigerator taking heat $Q_c^* - Q_c > 0$ into the engine and with no work input, puts heat $Q_h^* - Q_h > 0$ into the hot reservoir. But this is illegal according to Clausius statement of the second law. Hence e can only be less than or equal to e_c , proving Carnot's theorem.

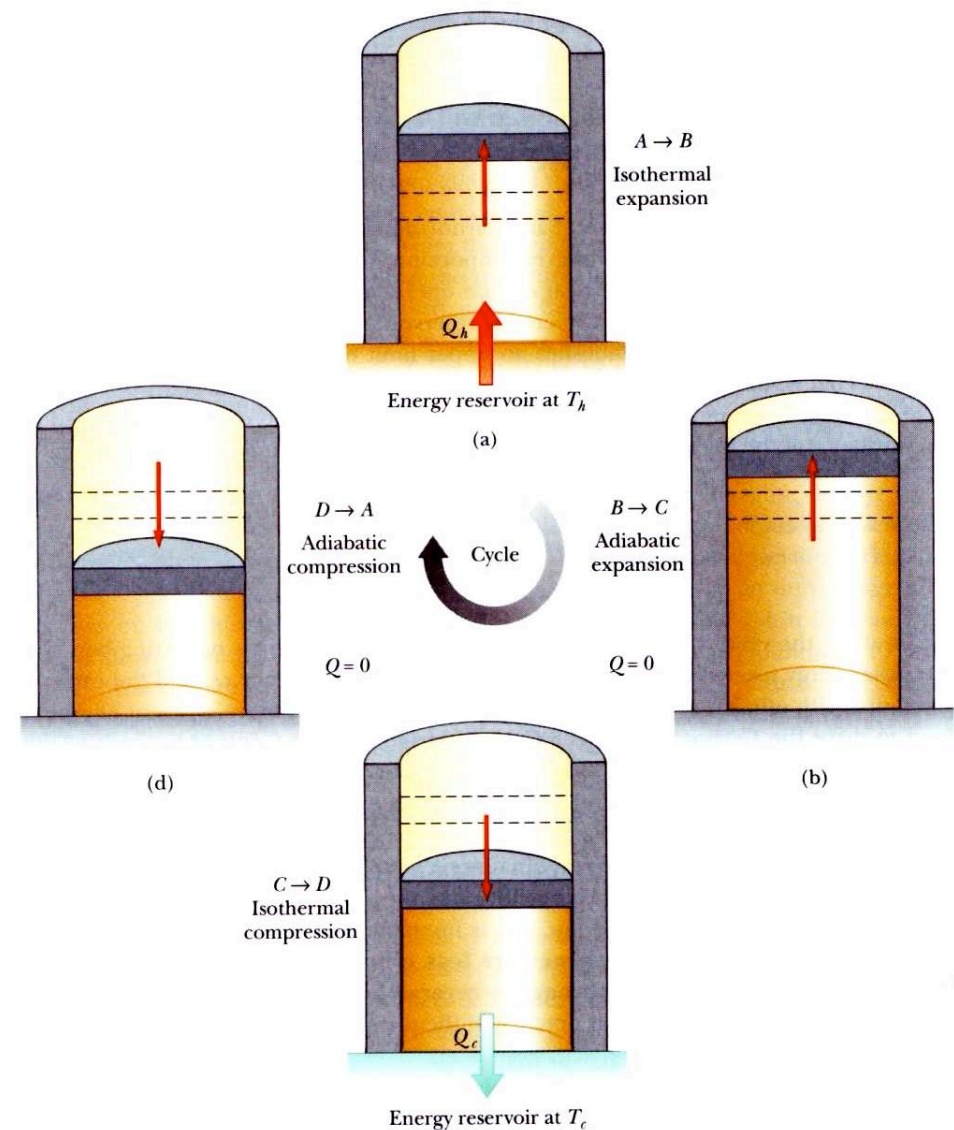


On top of that, it might also be clear why some people also call it Carnot's statement of the second law. Even though it doesn't talk about the second law specifically, it is as intimately linked to it as the Kelvin-Planck and Clausius statements are.



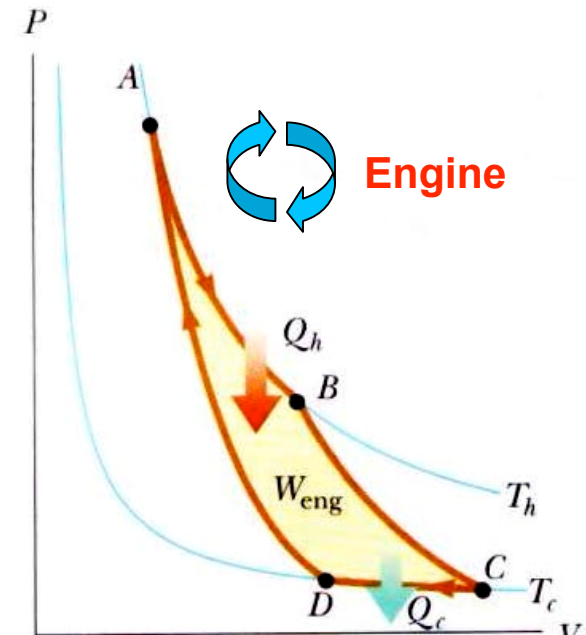
The four steps of the Carnot cycle

- The Carnot engine consists of two reservoirs – a hot reservoir at T_h and a cold reservoir at T_c – and a piston and cylinder containing an ideal gas. The cylinder is made from some magical material that is a perfect insulator during the adiabatic processes and an excellent heat conductor during the isothermal processes. The alternative is that the piston is always an insulator, in which case we do the adiabatic processes very quickly, but the isothermal processes take nearly forever.



The four steps of the Carnot cycle

- The four steps of the process and the corresponding PV -cycle are shown below.
1. The first process $A \rightarrow B$ is an isothermal expansion at T_h with the piston in thermal contact with the hot reservoir. During the expansion, the gas absorbs heat Q_h from the hot reservoir and does work W_{AB} in raising the piston.
 2. Process $B \rightarrow C$ is an adiabatic expansion, so no heat enters or leaves the system. During the expansion the internal energy changes and the temperature of the gas decreases from T_h to T_c . The gas does work W_{BC} in raising the piston.
 3. Process $C \rightarrow D$ is an isothermal compression at T_c with the piston in thermal contact with the cold reservoir. During the compression, the gas expels heat Q_c to the cold reservoir and work W_{CD} is done on the gas through the piston.
 4. Process $D \rightarrow A$ is an adiabatic compression, so no heat enters or leaves the system. During the compression the internal energy changes and the temperature of the gas increases from T_c to T_h . Work W_{DA} is done on the gas through the piston.



The efficiency of the Carnot cycle

- Because it's a closed cycle, the total change in internal energy ΔU is zero, the work done is the area enclosed in the PV -cycle, and is equal to $W = Q_h - Q_c$. As we know from lecture 16:

$$e = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h} \quad (10.2/3)$$

But for the Carnot cycle we can take this further. Firstly, we need to work out Q_h and Q_c and these come from the two isothermal processes in the cycle A-B and C-D. Given an isothermal process has $\Delta U = 0$, this means $Q = -W$ and so we can write:

$$Q_{AB} = -W_{AB} = \int_{V_A}^{V_B} \frac{nRT_h}{V} dV = nRT_h \ln\left(\frac{V_B}{V_A}\right) \quad (14.1)$$

and

$$Q_{CD} = -W_{CD} = \int_{V_C}^{V_D} \frac{nRT_c}{V} dV = nRT_c \ln\left(\frac{V_D}{V_C}\right) = -nRT_c \ln\left(\frac{V_C}{V_D}\right) \quad (14.2)$$

Note the signs carefully: $V_B > V_A$ so Q_{AB} is positive and it's a heat flow into the system, as we'd expect (it's an expansion), and $V_C > V_D$ so Q_{CD} is negative and it's a heat flow out of the system (it's a compression).



The efficiency of the Carnot cycle

- Now Q_h is the heat flowing out of the hot reservoir into the engine, and so $Q_h = Q_{AB}$, but Q_c is the heat flowing into the cold reservoir from the engine, which has the opposite sign to the heat flowing into the system at the cold reservoir side Q_{CD} , hence $Q_c = -Q_{CD}$. So:

$$Q_h = nRT_h \ln\left(\frac{V_B}{V_A}\right) \quad \text{and} \quad Q_c = -nRT_c \ln\left(\frac{V_C}{V_D}\right) \quad (14.3)$$

We can now put Eqn. 14-3 into Eqn. 10-2/3, which gives:

$$e = 1 - \frac{nRT_c \ln(V_C/V_D)}{nRT_h \ln(V_B/V_A)} \quad (14.4)$$

To take this further, we need to consider the two adiabatic processes for a moment. Considering the two side by side, we get:

$$T_C V_C^{\gamma-1} = T_B V_B^{\gamma-1} \quad T_D V_D^{\gamma-1} = T_A V_A^{\gamma-1} \quad (14.5)$$

Noting that $T_A = T_B = T_h$ and $T_C = T_D = T_c$, then:

$$T_c V_C^{\gamma-1} = T_h V_B^{\gamma-1} \quad T_c V_D^{\gamma-1} = T_h V_A^{\gamma-1} \quad (14.6)$$



The efficiency of the Carnot cycle

- If we now divide Eqn 14-6(a) by Eqn 14-6(b), we get:

$$\frac{T_c V_C^{\gamma-1}}{T_c V_D^{\gamma-1}} = \frac{T_h V_B^{\gamma-1}}{T_h V_A^{\gamma-1}} \quad (14.7)$$

The temperatures cancel out, and if you take both sides to the power of $1/(\gamma-1)$:

$$\frac{V_C}{V_D} = \frac{V_B}{V_A} \quad \text{or} \quad \ln\left(\frac{V_C}{V_D}\right) = \ln\left(\frac{V_B}{V_A}\right) \quad (14.8)$$

We can now use Eqn. 14-8(b) to cancel out the volume terms in Eqn. 14-4, so:

$$e = 1 - \frac{T_c}{T_h} = e_C \quad (14.9)$$

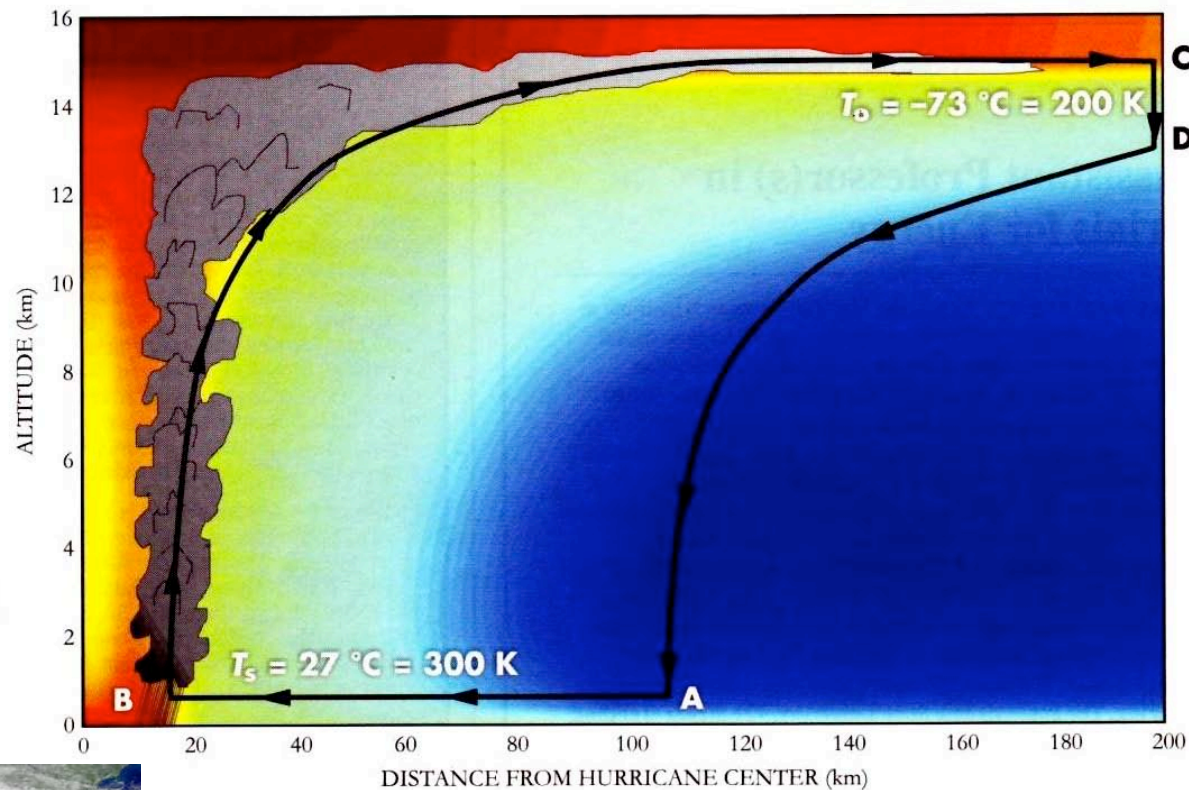
This is a key result, because the efficiency of a Carnot engine has nothing to do with the engine itself, nor how much heat flows in or out of it, nor how much work is done. The efficiency depends entirely on the temperature of the two reservoirs that the Carnot engine is attached to, that's all! The higher the temperature difference, the higher the efficiency, and if $T_c = T_h$, then $e_C = 0$ (i.e., our engine stops working). This is not the case for irreversible engines because $e < e_C$.



Can anyone spot the Carnot Engine?

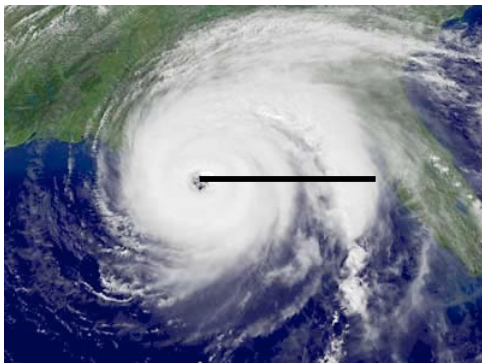


A real life Carnot cycle



The hurricane as a Carnot heat engine. This two-dimensional plot of the thermodynamic cycle shows a vertical cross section of the hurricane, whose storm center lies along the left edge. Colors depict the entropy distribution; cooler colors indicate lower entropy. The process mainly responsible for driving the storm is the evaporation of seawater, which transfers energy from sea to air. As a result of that transfer, air spirals inward from A to B and acquires entropy at a constant temperature. It then undergoes an adiabatic expansion from B to C as it ascends within the storm's eyewall. Far from the storm center, symbolically between C and D, it exports IR radiation to space and so loses the entropy it acquired from the sea. The depicted compression is very nearly isothermal. Between D and A the air undergoes an adiabatic compression. Voilà, the four legs of a Carnot cycle.

within the storm's eyewall. Far from the storm center, symbolically between C and D, it exports IR radiation to space and so loses the entropy it acquired from the sea. The depicted compression is very nearly isothermal. Between D and A the air undergoes an adiabatic compression. Voilà, the four legs of a Carnot cycle.



Our first look at entropy

- Let's take a step back through the mathematics, because I want to hit a result that we're going to discuss in a lot more detail in the next two lectures. Let's start at Eqn. 14-3 and divide Eqn. 14-3 (b) by Eqn. 14-3(a) to get:

$$\frac{Q_c}{Q_h} = \frac{nRT_c \ln(V_C/V_D)}{nRT_h \ln(V_B/V_A)} \quad (14.10)$$

If we now use our result from Eqn. 14-8(b) again to eliminate the volume terms:

$$\frac{Q_c}{Q_h} = \frac{T_c}{T_h} \quad (14.11)$$

we can rearrange this as:

$$\frac{Q_c}{T_c} = \frac{Q_h}{T_h} \quad (14.12)$$

Eqn. 14-12 is actually saying something quite profound. It's not that $Q_c = Q_h$ because in that case we'd get no work from our engine. And it's not that $T_c = T_h$ because then the engine doesn't work at all. What it's saying is that **the ratio Q/T is conserved** in a Carnot engine, and this ratio is something that is called the Entropy S . Hence the Carnot cycle is often called an *isentropic* (i.e., constant entropy) process.



Advanced – The (Re)definition of temperature

- We earlier defined temperature as ‘the thing that’s equal when two bodies are at thermal equilibrium’ and for a monatomic ideal gas at least, we showed that it’s equal to the mean kinetic energy of the particles divided by $3/2 k_B$.

Otherwise temperature is kind of arbitrary, for example with the Celsius scale we make T linear and set 0°C as the melting point of ice and 100°C as the boiling point of water, whereas with the Fahrenheit scale, it’s something else entirely.

A really nice aspect of the Carnot cycle is that we can use it to define temperature independent of the system or substance!



Advanced – The (Re)definition of temperature

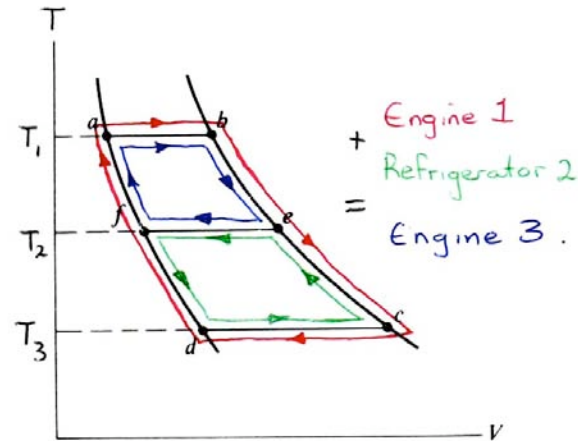
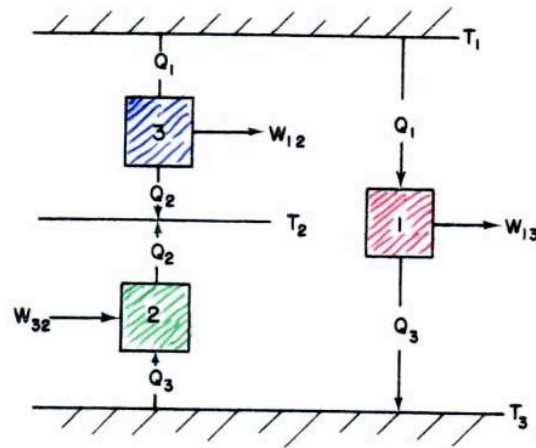


Fig. 5-2 Carnot cycles represented in the T - V plane. Curves a - f - d and b - e - c are reversible adiabatics.

- Consider the engine combination above, we have three reversible engines and three reservoirs at T_1 , T_2 and T_3 . Engine 1 and refrigerator 2 together are equivalent to engine 3, such that:

$$W_{13} - W_{32} = (Q_1 - Q_3) - (Q_2 - Q_3) = Q_1 - Q_2 = W_{12} \quad (14.13)$$

In this system, we can always relate the heat absorbed at T_1 to the heat delivered at T_2 by finding the heat delivered at some other temperature T_3 . Hence we can get all the engines' properties if we introduce a standard temperature to replace T_3 , analysing everything with respect to that standard temperature, because we can work from T_1 down to the standard temperature T_3 back up to T_2 .

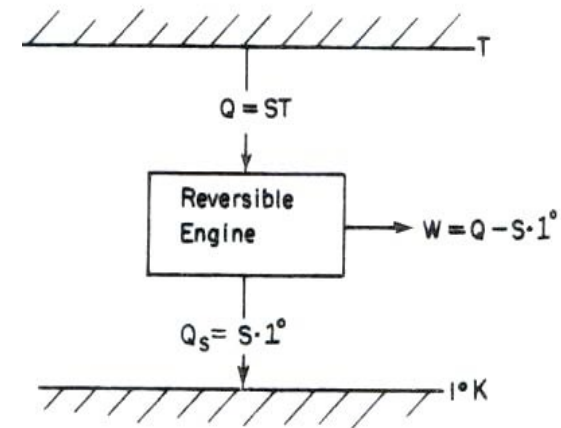


Advanced – Absolute thermodynamic temperature

- We can define the standard temperature arbitrarily as *one degree* (1°) and the heat delivered at this temperature as Q_S . And so all we really have to do is to find how much heat Q_1 we need to put in at temperature T_1 in order to deliver a certain amount of heat Q_S to our reservoir at 1° , because:

$$\frac{Q_1}{Q_S} = \frac{T_1}{1^\circ} \quad \text{so} \quad T_1 = \frac{Q_1}{Q_S} 1^\circ \quad (14.14)$$

This means we can tell how hot an object is by finding out how much heat is absorbed by a reversible engine working between the temperature of that object and 1° . For example, if seven times more heat is taken out of a boiler than is delivered at the 1° condenser, the temperature of the boiler is called 7° , and so forth.



The key point here is that we can make a definition of temperature which is independent of any particular substance, and for this reason the temperature defined in this way is called the *absolute thermodynamic temperature*.



Summary

- Since the Kelvin-Planck statement says we can't have a 100% efficient engine, Carnot asked the question of what is the most efficient engine that physics will allow, if we put all practical and engineering issues aside.
- The Carnot cycle consists of two isothermal processes and two adiabatic processes. It represents the most efficient engine possible because it is perfectly reversible. As a theoretical engine it represents the ultimate, but it doesn't work practically because all real processes are irreversible.
- Carnot's theorem states that no real engine can be more efficient than a Carnot engine operating between the same two reservoirs and that all Carnot engines operating between reservoirs at the same temperature have the same efficiency.
- Carnot's theorem can be proven by considering a more efficient 'super-engine' driving a Carnot refrigerator (i.e., a Carnot engine running backwards), and showing that the resulting combination violates the Clausius statement of the Second law.
- The efficiency of a Carnot engine can be determined as $e_c = 1 - T_c/T_h$. Furthermore, it can be shown that the entropy $S = Q/T$ is conserved in a Carnot engine.

In the next lecture we will look at entropy from the microscopic viewpoint to get a viewpoint for what it really is and how it fits in with the 2nd law of thermodynamics.

